

# CAPACITY OF AN ASSOCIATIVE MEMORY MODEL ON RANDOM GRAPH ARCHITECTURES

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**ABSTRACT.** We analyze the storage capacity of the Hopfield models on classes of random graphs. While such a setup has been analyzed for the case that the underlying random graph model is an Erdős-Renyi-graph, other architectures, including those investigated in the recent neuroscience literature, have not been studied yet. We develop a notion of storage capacity that highlights the influence of the graph topology and give results on the storage capacity for not too irregular random graph models. The class of models investigated includes the popular power law graphs for some parameter values.

## 1. INTRODUCTION

Thirty years ago, in 1982, Hopfield introduced a toy model for a brain that renewed the interest in neural networks and has nowadays become popular under the name Hopfield model [14]. This model in its easiest version assumes that the neurons are fully connected and have Ising-type activities, i.e. they take the values  $+1$ , if a neuron is firing and  $-1$ , if it is not, and is based on the principles of statistical mechanics. Since Hopfield's ground-breaking work it has stimulated a large number of researchers from the areas of computer science, theoretical physics and mathematics.

In the latter field the Hopfield model is particularly challenging, since it also can be considered as a spin glass model and spin glasses are notoriously difficult to study. A survey over the mathematical results in this area can be found in either [4] or [30]. It is worth mentioning, that even in the parameter region where no spin glass phase is expected the Hopfield model still has to offer surprising phenomena such as in [13].

When being considered as a neural network one of the aspects that have been discussed most intensively is its so-called storage capacity. Here, one tries to store information, so-called patterns in the model, and the question is, how many patterns can be successfully retrieved by the network dynamics, i.e. how much information can be stored in a model of  $N$  neurons. One of the early mathematical

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results states, that if the patterns are independent and identically distributed (i.i.d. for short) and consist of i.i.d spins and if their number  $M$  is bounded by  $\frac{1}{2}N/\log N$ , the patterns can be recalled (see [24]) with probability converging to one as  $N \rightarrow \infty$  and that the constant  $\frac{1}{2}$  is optimal (see [3]). Similar results hold true, if one starts with a corrupted input – if more than fifty percent of the input spins are correct, one still is able to restore the originally "learned" patterns. However, if one also allows for small errors in the retrieval of the patterns one obtains a storage capacity of  $M = \alpha N$  for some value of  $\alpha$  smaller than 0.14 (see [25], [18], [29]). This latter result is in agreement with both, computer simulations as well as the predictions of the non-rigorous replica method from statistical physics (see [1]).

The setup of the Hopfield model has been generalized in various aspects, e.g. the condition of the independence has been relaxed (see [19], [22]), patterns with more than two spins values have been considered (see [12], [20], [21]), and Hopfield models on Erdős-Rényi-graphs were studied ([5], [6], [29], [23]). The present paper starts with the observation that, even though being more general than the complete graph, also Erdős-Rényi-graphs do not seem to be the favorite architectures for a brain for scientists working in neurobiology. There, the standard paradigm currently is rather to model the brain as a small world graph (see [2], [28]). We will focus on the question, how many patterns can be stored in a Hopfield model on a random graph, if this graph is no longer necessarily an Erdős-Rényi-graph. There is a major influence of the graph structure on the model's capability to retrieve corrupted information. The relationship between network connectivity and the performance of associative memory models has already been investigated in computer simulations (see for instance [7]). Therefore the goal of the present note is to establish rigorous bounds on the storage capacity of the Hopfield model on a wide class of random graph models, where we interpret "storage" as the ability to retrieve corrupted information.

To this end we organize the paper in the following way: Section 2 introduces the basic model we will be working with in the present paper. It also addresses the question, what exactly we mean when talking about the storage of patterns. Section 3 contains the main result of this paper. A main ingredient of the proof is to analyze the spectrum of the adjacency matrix of the graph that serves as a model of the network architecture. This analysis is provided in Section 4. Eventually, Section 5 contains the proof of the main result. An appendix will contain estimates on the minimum and maximum degree of an Erdős-Rényi graph. These are needed to apply our main result to the setting of such random graphs and may also be of independent interest.

## 2. THE MODEL

The Hopfield model is a spin model on  $N \in \mathbb{N}$  spins.  $\sigma \in \Sigma_N := \{-1, +1\}^N$  describes the neural activities of  $N$  neurons. The information to be stored in the model are patterns  $\xi^1, \dots, \xi^M \in \{-1, +1\}^N$ . As usual, we will assume that these patterns are i.i.d. and consist of i.i.d. spins  $(\xi_i^\mu)$  with

$$\mathbb{P}(\xi_i^\mu = \pm 1) = \frac{1}{2}.$$

Note that  $M$  may and in the interesting cases will be a function of  $N$ . The architecture of the Hopfield model is an undirected graph  $G = (V, E)$ , where  $V = 1, \dots, N$ . With the help of the patterns and the graph one defines the Hamiltonian (or energy function) of the model by

$$H_N(\sigma) = -\text{Const.}(N) \sum_{i,j=1}^N \sigma_i \sigma_j a_{ij} \sum_{\mu=1}^M \xi_i^\mu \xi_j^\mu.$$

Here  $a_{ij} = a_{ji} = 1$  if the edge between  $i$  and  $j$  is in  $E$  and  $a_{ij} = a_{ji} = 0$  otherwise. The constant is chosen in such a way that the mean free energy of the model is finite and not constantly equal to zero. The idea of this setup is that the patterns (as well as their negatives  $-(\xi^\mu), \mu = 1, \dots, M$ ) are hopefully local minima of  $H_N$ . This is easily checked, if  $M \equiv 1$  and  $G$  is the complete graph, since then

$$H_N(\sigma) = -\text{Const.}(N) \left( \sum_{i=1}^N \sigma_i \xi_i^1 \right)^2$$

and hoped to be inherited by the more general model, as long as  $M$  is small enough. If this is indeed the case, then not only the patterns are points in  $\Sigma_N$  that have a large weight under the (random) Gibbs measure

$$\mu_N(\sigma) := \frac{\exp(-\beta H_N(\sigma))}{Z_N}, \quad \beta > 0$$

(where  $Z_N$  is a normalizing constant), but also they are fixed points of the parallel gradient descent dynamics  $T = (T_i)$  on  $\Sigma_N$ . By definition  $T_i$  only changes the  $i$ 'th coordinate of a configuration  $\sigma$  and

$$T(\sigma) = (T_1(\sigma), \dots, T_N(\sigma)), \text{ with } T_i(\sigma) = \text{sgn}\left(\sum_{j=1}^N \sigma_j a_{ij} \sum_{\mu=1}^M \xi_i^\mu \xi_j^\mu\right).$$

Hence  $T_i$  flips the  $\sigma_i$  if and only if this lowers the energy. The dynamics  $T$  can be thought of as governing the evolution of the system from an input towards the nearest learned pattern.  $\xi^\mu$  being fixed point of  $T$  can thus be interpreted as recognizing a learned pattern. However, this is not really what one would call an associative memory. An important feature of the standard Hopfield model (the one where  $G = K_N$ , the complete graph on  $N$  vertices) is thus also that under certain restrictions on  $M$  (and the number of corrupted neurons), with

high probability, a corrupted version of  $\xi^\mu$ , say  $\tilde{\xi}^\mu$  converges to  $\xi^\mu$  when being evolved under  $T$ . This observation is also crucial for the present paper.

When considering the stability of a random pattern  $\xi^\mu$  under  $T$  in the above setting, we need to check whether  $T_i(\xi^\mu) = \xi_i^\mu$  holds for any  $i$ . Now

$$T_i(\xi^\mu) = \text{sgn}\left(\sum_{j=1}^N a_{ij} \sum_{\nu=1}^M \xi_i^\mu \xi_j^\mu \xi_j^\nu\right) = \text{sgn}\left(\sum_{j=1}^N a_{ij} \xi_i^\mu + \sum_{j=1}^N a_{ij} \sum_{\nu \neq \mu} \xi_i^\mu \xi_j^\mu \xi_j^\nu\right).$$

I.e. we have a signal term of strength  $d(i)$  (the degree of vertex  $i$ ) and a random noise term. The first observation is that the network topology enters via the degrees of the nodes. Indeed in such a simple setup – the stability of stored information – the minimum degree of the vertices is clearly decisive to compute the models's storage capacity : in the case where a vertex  $i$  has a small degree, the noise term will exceed the signal term, except for a very small number of stored patterns. However, it seems to be obvious that also global aspects, e.g. whether or not the graph is connected, must play a role. This is confirmed, if we are setting up a Hopfield model on graph  $G$  consisting of a complete graph  $K_m$  (on the vertices  $1, \dots, m$ ) and the graph  $K_{N-m}$  on the vertices  $m+1, \dots, N$  with  $\log N \ll m \ll N$  and if we assume that these two subgraphs are disconnected or just connected by one arc. Each of the vertices thus has at least degree  $m$  and it can be computed along the lines of [24] or [26] that at least  $\frac{m}{2 \log N}$  patterns can be stored as fixed points of the dynamics  $T$ . However, if we try to store one pattern, e.g.  $\xi^1$  with  $\xi_i^1 = 1$  for all  $i = 1, \dots, N$ , and start with a corrupted input  $\tilde{\xi}^1$  with

$$\tilde{\xi}_i^1 = \begin{cases} -1 & i \leq m \\ 1 & m+1 \leq i \leq N \end{cases}$$

we see that

$$T_i(\tilde{\xi}^1) = \tilde{\xi}_i^1.$$

Hence  $\tilde{\xi}^1$  is a fixed point of  $T$  implying that the retrieval dynamics is not able to correct  $m \ll N$  errors, even if we just want to store one pattern. So, if we insist that a neural network should also exhibit some associative abilities (and this has always been a central argument for the use of neural networks), we have to take the graph topology into account.

This topology is encoded in the so called adjacency matrix  $A$  of  $G$ . Here  $A = (a_{ij})$  and  $a_{ij} = 1$ , if  $e_{i,j} \in E$  and  $a_{ij} = 0$  otherwise. If  $G$  is sufficiently regular, the connectivity of  $G$  (which played an important role in the above counterexample) can be characterized in terms of the spectral gap. To define it, let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$  be the (necessarily real) eigenvalues of  $A$  in decreasing order. Define  $\kappa$  to be the second largest modulus of the eigenvalues, i.e.

$$\kappa := \max_{i \geq 2} |\lambda_i| = \max\{\lambda_2, |\lambda_N|\}.$$

Then the spectral gap is the difference between the largest eigenvalue and  $\kappa$ , i.e.  $\lambda_1 - \kappa$ . However, also the degrees of the vertices are important. Hence let

$d_i = \sum_j a_{ij}$  be the degree of vertex  $i$ . We will denote by

$$\delta := \min_i d_i \quad \text{and} \quad m := \max_i d_i$$

the minimum and maximum degree of  $G$ , respectively.

### 3. RESULTS

We will now state the main result of the present paper.

In order to formulate it let us define the usual Hamming distance on the space of configurations  $\Sigma_N$ ,

$$d_H(\sigma, \sigma') = \frac{1}{2}[N - (\sigma, \sigma')].$$

In other words,  $d_H$  counts of indices where  $\sigma$  and  $\sigma'$  disagree. For any  $\sigma \in \Sigma_N$  and  $\varrho \in [0, 1]$ , let  $\mathcal{S}(\sigma, \varrho)$  the sphere of radius  $\varrho N$  centered at  $\sigma$ , i.e.

$$\mathcal{S}(\sigma, \varrho) = \{\sigma' : d_H(\sigma, \sigma') = [\varrho N]\},$$

where  $[\varrho N]$  denotes the integer part of  $\varrho N$ .

For the rest of the paper we will suppose that the following hypothesis is true :

(H<sub>1</sub>) There exists  $c_1 \in ]0, 1[$ , such that  $\delta > c_1 \lambda_1$  (recall that  $\delta$  is the minimum degree of the graph  $G$ , and  $\lambda_1$  is the largest eigenvalue of its adjacency matrix).

To understand condition (H<sub>1</sub>) recall that for a regular graph with degree  $d$  the largest eigenvalue of  $A$  equals  $d$  and so does its minimum degree  $\delta$ . Condition (H<sub>1</sub>) can thus be interpreted as the requirement that  $G$  is sufficiently regular.

Under this condition we will prove, that we can store a number  $M$  of patterns depending on  $\lambda_1$  and the spectral gap of  $A$  – even in the sense that the dynamics  $T$  repairs a corrupted input. Mathematically speaking we show

**Theorem 3.1.** *With the notation introduced in Section 2, if (H<sub>1</sub>) is satisfied and  $\lambda_1 \geq c \log(N) \kappa$  for some  $c > 0$  large enough, then there exists  $\alpha_c > 0$  and  $\rho_0 \in ]0, 1/2[$  such that if*

$$M = \alpha \frac{\lambda_1^2}{m \log N} - \frac{\kappa \lambda_1}{m},$$

*for some  $\alpha < \alpha_c$ , then that for all  $\rho \in ]0, \rho_0]$  we obtain*

$$P[\forall \mu = 1, \dots, M, \forall x \in \mathcal{S}(\xi^\mu, \rho) : T^k(x) = \xi^\mu] \rightarrow 1, \text{ as } N \rightarrow \infty,$$

*for  $k = \mathcal{O}(\max\{\log \log N, \frac{\log(N)}{\log(\frac{\lambda_1}{\kappa \log(N)})}\})$ . Here  $T^k$  is defined as the  $k$ 'th iterate of the map  $T$ .*

In other words, Theorem 3.1 states that we are able to store the given number of patterns in such a way, that a number of errors that is proportional to  $N$  can be repaired by a modest (at most  $\mathcal{O}(\log N)$ ) number of iterations of the retrieval dynamics. The number of patterns depends on the largest eigenvalue and the spectral gap of the adjacency matrix and is larger for large spectral gaps. Before advancing to the proof we will apply this result to some classical models of random and non-random graphs.

**Corollary 3.2.** *If  $G = K_N$ , i.e. in the case of the classical Hopfield model, the storage capacity in the sense of Theorem 3.1 is  $M = \alpha \frac{N}{\log N}$  for some constant  $\alpha$ . The number of steps needed to repair a corrupted input is of order  $\mathcal{O}(\log \log N)$ .*

*Proof.* The complete graph is regular, hence condition (H1) is satisfied. From Theorem 3.1 we obtain the numerical values for  $M$  and the number of steps by observing that in the case of the complete graph the eigenvalues of  $A$  are  $N - 1$  and  $-1$  (the latter being an  $N - 1$ -fold eigenvalue).  $\square$

It should be remarked that similar results were obtained by Komlos and Paturi [16].

We mainly want to apply our results to some random architectures, i.e.  $G$  will be the realization of some random graph. The most popular model of a random graph is the Erdős-Renyi-graph  $G(N, p)$ . Here all the possible  $\binom{N}{2}$  edges occur with equal probability  $p = p(N)$  independently of each other. Hopfield models on  $G(N, p)$  have already been discussed in [5], [29], or [23].

Here we obtain

**Corollary 3.3.** *If  $G$  is chosen randomly according to the model  $G(N, p)$ , then, if  $p \gg \frac{\log N}{N}$  for a set of realizations of  $G$  the probability of which converges to one as  $N \rightarrow \infty$ , the capacity (in the above sense) of the Hopfield model is  $c pN / \log(N)$  for some constant  $c > 0$ .*

*Proof.* For the eigenvalues of an Erdős-Rényi-graph it is well known that with probability converging to 1, as  $N \rightarrow \infty$  (such a statement in random graph theory is said to hold asymptotically almost surely),  $\lambda_1 = (1 + o(1))Np$  and  $\kappa \leq c\sqrt{Np}$  (see for instance [23]). Moreover, we can control the minimum and maximum degree in  $G(N, p)$ . Indeed, for our values of  $p$  we have  $m = (1 + o(1))Np$  and  $\delta = (1 + o(1))Np$  asymptotically almost surely. Surprisingly, we could not find this result in the literature and thus proved it in Appendix A.

Hence  $(H_1)$  is satisfied.  $\square$

For the next example let us give a general construction of random graph models. To this end let  $i_0$  and  $N$  positive integers and  $L = \{i_0, i_0 + 1, i_0 + N - 1\}$ . For a sequence  $w = (w_i)_{i \in L}$ , we consider random graphs  $G(w)$  in which edges are assigned independently to each pair of vertices  $(i, j)$  with probability

$$p_{ij} = \rho w_i w_j,$$

where  $\rho = 1/\sum_{k \in L} w_k$ . We assume that

$$\max_i w_i^2 < \sum_{k \in L} w_k$$

so that  $p_{ij} \leq 1$  for all  $i$  and  $j$ . It is easy to see that the expected degree of  $i$  is  $w_i$ . For notational convenience let  $d = \sum_{i \in L} w_i/N$  be the expected average degree,  $\overline{m}$  the expected maximum degree and

$$\tilde{d} = \sum_{i \in L} w_i^2 / \sum_{i \in L} w_i$$

be the so called second order average degree of the graph  $G(w)$ .

We now turn to a subclass of random graphs that have recently become very popular, power law graphs [11]. Power law random graphs are random graphs in which the number of vertices of degree  $k$  is proportional to  $1/k^\beta$  for some fixed exponent  $\beta$ . It has been realized that this "power law"-behavior is prevalent in realistic graphs arising in various areas. Graphs with power law degree distribution are ubiquitously encountered, e.g. in the Internet, the telecommunications graphs, the neural networks and many biological applications ([15], [27],[31]). Using the  $G(w)$  model, we can build random power law graphs in the following way. Given a power law exponent  $\beta$ , a maximum expected degree  $\overline{m}$ , and an average degree  $d$ , we take  $w_i = ci^{-\frac{1}{\beta-1}}$  for each  $i \in \{i_0, \dots, i_0 + 1, i_0 + N - 1\}$ , with

$$c = \frac{\beta - 2}{\beta - 1} d N^{\frac{1}{\beta-1}}$$

and

$$i_0 = N \left( \frac{d(\beta - 2)}{\overline{m}(\beta - 1)} \right)^{\beta-1}.$$

For such power law graphs we obtain:

**Corollary 3.4.** *If  $G$  is chosen randomly according to a power law graph with  $\beta > 3$  then, if  $\overline{m} \gg d > c\sqrt{\overline{m}}(\log(N))^{3/2}$  for some constant  $c > 0$  for a set of realizations of  $G$  the probability of which converges to one as  $N \rightarrow \infty$ , the capacity (in the above sense) of the Hopfield model is  $C(\beta) \frac{d^2}{\overline{m} \log(N)}$  for a constant  $C$  that only depends on  $\beta$ .*

*Proof.* By definition, if  $d \ll \overline{m}$ , then the minimum expected degree  $w_{\min} = c(i_0 + N - 1)^{-\frac{1}{\beta-1}}$  satisfies  $w_{\min} = \frac{\beta-2}{\beta-1} d(1 + o(1))$ .

From [8], we learn about the second order average degree that

$$\tilde{d} = (1 + o(1)) \frac{(\beta - 2)^2}{(\beta - 1)(\beta - 3)} d,$$

if  $\beta > 3$ .

On the other hand, Chung and Radcliffe prove in [10] the following : if the maximum expected degree  $\overline{m}$  satisfies  $\overline{m} > \frac{8}{9} \log(\sqrt{2}N)$ , then with probability at



least  $1 - \frac{1}{N}$ , we have

$$\lambda_1(G(w)) = (1 + o(1))\tilde{d} \quad \text{and} \quad \kappa(G(w)) \leq \sqrt{8\overline{m} \log(\sqrt{2}N)}.$$

From this, in particular, we can deduce that hypothesis  $(H_1)$  is satisfied.

To apply Theorem 3.1 we need to compare  $\overline{m}$  to the maximum degree  $m$  of a graph  $G$ , chosen randomly according to a power law graph with  $\beta > 3$ . As proven in [9], we have the following estimate, using Chernoff inequalities : for all  $c > 0$ , there exist two constants  $c_0, c_1 > 0$  such that

$$P[\exists i \in L : |d_i - w_i| > cw_i] \leq \sum_{i \in L} \exp(-c_0 w_i) \leq \exp(-c_1 d + \log N),$$

since  $w_{\min} = \mathcal{O}(d)$ .

Under our assumption that  $d \gg \log N$ , we deduce from this estimate that  $m = C \overline{m}(1 + o(1))$ , for some  $C > 0$ , and we finally obtain that the capacity of the Hopfield model on power law graphs (for a sequence of sets of graphs with probability converging to one) is at least

$$\text{const} \frac{\lambda_1^2}{m \log(N)} - \frac{\kappa \lambda_1}{m} = C(\beta) \frac{d^2}{\overline{m} \log(N)},$$

if,  $\beta > 3$ , and  $\kappa < c_2 \frac{\lambda_1}{\log(N)}$  for some  $c_2 > 0$  small enough. This is true in particular, if

$$\sqrt{8\overline{m} \log(\sqrt{2}N)} < c_3 \frac{d}{\log(N)}$$

for some  $c_3$  small enough, i.e.

$$d > c\sqrt{\overline{m}}(\log(N))^{3/2}.$$

□

#### 4. TECHNICAL PREPARATIONS ON RANDOM GRAPHS

We first present the results we will use in the proof of our theorem. Let  $G$  be a simple graph with  $N$  vertices and  $l$  edges. Recall that for such a graph

$$\lambda_1 \geq \dots \geq \lambda_N$$

are the (real) eigenvalues of its adjacency matrix and  $\kappa = \max\{\lambda_2, |\lambda_N|\}$ .

We begin with an estimate of the moment generating function of a sum of i.i.d. random variables, related to  $G$ . We assign i.i.d. random variables  $X_i$  to the vertices of  $G$ , taking values  $\pm 1$  with equal probability. Let us define the "quadratic form" over  $G$

$$S = \sum_{\{i,j\} \in E} X_i X_j.$$

The following theorem due to Komlos and Paturi gives an upper bound on the moment generating function of  $S$ , which appears naturally when we use an exponential Markov inequality for an upper bound.



**Theorem 4.1** (J. Komlos, R. Paturi [17]). *The moment generating function of  $S$  can be bounded as*

$$E[e^{-tS}] \leq E[e^{tS}] \leq \exp\left(\frac{lt^2}{2(1 - \lambda_1 t)}\right),$$

for  $0 \leq t < 1/\lambda_1$ .

**Corollary 4.2.** *For any  $y > 0$ , we have*

$$P[S > y] \leq \exp\left(-\frac{y^2}{2(l + \lambda_1 y)}\right).$$

*Proof.* Apply the exponential Markov inequality together with Theorem 4.1 to see that

$$P[S > y] \leq e^{-ty} E[e^{tS}] \leq \exp\left(-ty + \frac{lt^2}{2(1 - \lambda_1 t)}\right),$$

for  $0 \leq t < 1/\lambda_1$ . The desired estimate is obtained by the choice of  $t = \frac{y}{l + \lambda_1 y}$  which is smaller than  $1/\lambda_1$ .  $\square$

As we will apply this result for subgraphs in the proof of our main result, we need also an estimate of the largest eigenvalue  $\lambda_1(H)$  of particular subgraphs  $H$  of  $G$ . To this end we will quote another result by Komlos and Paturi.

**Lemma 4.3** (J. Komlos, R. Paturi [17]). *Let  $G$  be a simple graph with  $N$  vertices. If  $I$  and  $J$  are two subsets of the vertex set of  $G$  with  $|I| = \rho N$  and  $|J| = \rho' N$ , where  $\rho, \rho' \in (0, 1)$ . Then the number of directed edges  $e(J; I)$  going from  $J$  to  $I$  is at most*

$$e(J, I) \leq [\rho\rho'\lambda_1(G) + \sqrt{\rho\rho'}\kappa]N.$$

*Moreover the largest eigenvalue (of the adjacency matrix) of the graph  $H$  determined by the undirected edges from  $I$  to  $J$  is bounded as*

$$\lambda_1(H) \leq 2[\sqrt{\rho\rho'}\lambda_1(G) + (1 - \sqrt{\rho\rho'})\kappa(G)].$$

## 5. PROOF OF THE MAIN RESULT

We are now ready to begin with the Proof of Theorem 3.1. We first present an important lemma that determines the behavior of the system for one step of the synchronous dynamics, more precisely it controls, how many errors are corrected by one step of the dynamics.

**Lemma 5.1.** *Let*

$$\rho_m = \exp\left(-c_2 \frac{\lambda_1}{\kappa + \frac{Mm}{\lambda_1}}\right).$$

*If  $M \leq c\lambda_1$  for some constant  $c > 0$ , there exists  $\rho_0 \in (0, \frac{1}{2})$  and constants  $c_1, c_2 > 0$ , such that for all  $\rho \in [\rho_m, \rho_0]$  we have:*

$$P[\forall \mu \in \{1, \dots, M\}, \forall x \in \mathcal{S}(\xi^\mu, \rho) : d_H(T(x), \xi^\mu) \leq f(\rho) N] \geq 1 - \varepsilon_N,$$

where

$$f(\rho) = \max\{c_1\rho \left(\frac{\kappa}{\lambda_1}\right)^2, c_1\rho h(\rho), c_1\frac{\kappa}{\lambda_1}h(\rho), c_1\rho \left(\frac{M\kappa}{(\lambda_1)^2} \log\left(\frac{1}{\rho}\right)\right)^{2/3}, \rho_m\} \leq \rho,$$

$\varepsilon_N \geq 0$ ,  $\varepsilon_N \rightarrow 0$  as  $N \rightarrow +\infty$  and

$$h(\rho) = -\rho \log \rho - (1 - \rho) \log(1 - \rho)$$

is the entropy function.

*Proof.* We will prove that

$$\sum_{\mu=1}^M P[\exists x \in \mathcal{S}(\xi^\mu, \rho) : d_H(T(x), \xi^\mu) > f(\rho)N] \leq \varepsilon_N \quad (5.1)$$

To simplify notations, we can assume that the fundamental memory in question is  $\xi^1$ . Let  $x \in \{-1, 1\}^N$  such that  $d_H(\xi^1, x) = \rho N$  and  $I$  be the set of coordinates in which  $x$  and  $\xi^1$  differ. Let  $T(x)$  be the vector resulting after one step of the parallel dynamics, and  $J$  be the set of coordinates in which  $T(x)$  and  $\xi^1$  differ. Now define the weight matrix  $W$  as  $W = (w_{ij})$  and

$$w_{ij} = a_{ij} \sum_{\nu=1}^M \xi_i^\nu \xi_j^\nu.$$

Then for all  $j \in J$ , we have  $\xi_j^1(Wx)_j \leq 0$ , which implies  $\sum_{j \in J} \xi_j^1(Wx)_j \leq 0$ .

For later use set

$$S^\mu(J, I) = \sum_{j \in J} \sum_{k=1}^N a_{jk} \xi_j^1 \xi_j^\mu \xi_k^\mu x_k$$

and

$$S(J, I) = \sum_{\mu=1}^M S^\mu(J, I) =: \sum_{j \in J} \xi_j^1(Wx)_j.$$

Observe that if  $\rho < 1/2$ , then  $x$  tends to be closer to  $\xi^1$  than to any other pattern, and  $S^1(J, I)$  will be the dominating term in  $S(J, I)$ . We will first give a lower bound for  $S^1(J, I)$ . We can rewrite  $S^1(J, I)$  as

$$S^1(J, I) = \sum_{j \in J} \sum_{k=1}^N a_{jk} \xi_k^1 x_k = \sum_{j \in J} (e(j, \bar{I}) - e(j, I)) = e(J, V) - 2e(J, I)$$

where again we use the notation  $e(J, I)$  and  $e(j, I)$ , to denote the number edges going from the set  $J$  to the set  $I$ , or, respectively, from the vertex  $j$  to the set  $I$ .

Under the assumption of hypothesis  $(H_1)$  and with the help of Lemma 4.3, we have for all  $I$  and  $J$ ,

$$\begin{aligned} S^1(J, I) &\geq c_1 \lambda_1 |J| - 2(|I||J| \frac{\lambda_1}{N} + \sqrt{|I||J|} \kappa) \\ &= \lambda_1 |J| (c_1 - 2\rho - 2\sqrt{\frac{\rho}{\rho'}} \frac{\kappa}{\lambda_1}), \end{aligned}$$

where  $\rho' = \frac{|J|}{N}$ . If we assume that  $\rho' \geq c_2 \rho (\frac{\kappa}{\lambda_1})^2$  for some  $c_2 > 0$  large enough, and  $\rho < \rho_0$  for some  $\rho_0 \in (0, 1/2)$  small enough, we get

$$S^1(J, I) \geq C_1 \lambda_1 |J|, \quad (5.2)$$

for some constant  $C_1 \in (0, 1)$ .

For  $\mu \geq 2$ , we compute

$$\begin{aligned} S^\mu(J, I) &= \sum_{(j,k) \in E(J, \bar{I})} u_j^\mu u_k^\mu - \sum_{(j,k) \in E(J, I)} u_j^\mu u_k^\mu \\ &= \sum_{(j,k) \in E(J, V)} u_j^\mu u_k^\mu - 2 \sum_{(j,k) \in E(J, I)} u_j^\mu u_k^\mu, \end{aligned}$$

where  $u_i^\mu = \xi_i^1 \xi_i^\mu$ , for all  $i = 1, \dots, N$  and  $\mu = 1, \dots, M$ . To apply the results for the moment generating function of quadratic forms introduced in Theorem 4.1 and Corollary 4.2, we need to rewrite these sums over ordered pairs of vertices as sums over unordered pairs. We have

$$E(J, V) = E(J, J) + E(J, \bar{J}) = 2E\{J, J\} + E\{J, \bar{J}\} = E\{J, V\} + E\{J, J\},$$

where for  $K, L \subset V$   $E(K, L)$  is the edges set of the directed graph between the sets  $K$  and  $L$  induced by our original graph. Likewise  $E\{K, L\}$  denotes the corresponding set of undirected edges. In the same way we obtain:

$$\begin{aligned} E(J, I) &= E(J \cap \bar{I}, J \cap I) + E(J \cap I, J \cap I) + E(J, I \cap \bar{J}) \\ &= E\{J \cap \bar{I}, J \cap I\} + 2E\{J \cap I, J \cap I\} + E\{J, I \cap \bar{J}\} \\ &= E\{J, I\} + E\{J \cap I, J \cap I\} \end{aligned}$$

Eventually,

$$E(J, V) - 2E(J, I) = E\{J, V\} - 2E\{J, I\} + E\{J, J\} - 2E\{J \cap I, J \cap I\}.$$

We want to prove that

$$MP[\exists I, |I| = \rho N, \exists J, |J| = \rho' N, S(J, I) < 0] \longrightarrow 0,$$

as  $N \rightarrow +\infty$ .

To this end set

$$\begin{aligned} S_1^\mu(J) &= \sum_{(j,k) \in E\{J, V\}} u_j^\mu u_k^\mu, & S_2^\mu(J, I) &= \sum_{(j,k) \in E\{J, I\}} u_j^\mu u_k^\mu, \\ S_3^\mu(J) &= \sum_{(j,k) \in E\{J, J\}} u_j^\mu u_k^\mu, & \text{and } S_4^\mu(J, I) &= \sum_{(j,k) \in E\{J \cap I, J \cap I\}} u_j^\mu u_k^\mu. \end{aligned}$$

Then

$$S(J, I) = S^1(J, I) + \sum_{\mu=2}^M S_1^\mu(J) - 2 \sum_{\mu=2}^M S_2^\mu(J, I) + \sum_{\mu=2}^M S_3^\mu(J) - \sum_{\mu=2}^M S_4^\mu(J, I).$$

Let  $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0$ , such that  $\gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4 = 1$ .

We will consider the four sums separately. First, using (5.2), we have

$$\begin{aligned} & P[\exists I, |I| = \rho N, \exists J, |J| = \rho' N, \sum_{\mu=2}^M S_1^\mu(J) < -\gamma_1 S^1(J, I)] \\ & \leq \sum_{J: |J| = \rho' N} P[\sum_{\mu=2}^M S_1^\mu(J) < -\gamma_1 C_1 \lambda_1 |J|] \end{aligned}$$

Given the vector  $\xi^1 = (\xi_i^1)_{i=1, \dots, N}$ , the random variables  $(u_i^\mu)_{i=1, \dots, N}^{\mu=2, \dots, M}$  are conditionally independent and uniformly distributed on  $\{-1, +1\}$ . As the estimates we will get for the conditional probabilities and the moment generating function will not depend on the choice of  $\xi^1$ , they will be true also for the unconditional probabilities.

Given the vector  $\xi^1$ , the random variables  $S_1^\mu(J), \mu = 2, \dots, M$ , are independent. Similar to the estimate of Corollary 4.2, we obtain

$$P[\sum_{\mu=2}^M S_1^\mu(J) < -\gamma_1 C_1 \lambda_1 |J|] \leq \exp\left(-\frac{1}{2} \frac{\gamma_1 C_1 \lambda_1 |J|}{\lambda_J + \frac{Me\{J, V\}}{\gamma_1 C_1 \lambda_1 |J|}}\right),$$

where  $\lambda_J = \lambda_1(E\{J, V\})$  is the largest eigenvalue of the graph determined by the undirected edges in  $E\{J, V\}$ . Using Lemma 4.3, we have

$$\lambda_J \leq 2[\sqrt{\rho'} \lambda_1 + (1 - \sqrt{\rho'}) \kappa],$$

and moreover  $e\{J, V\} \leq e(J, V) \leq m|J|$  is trivially true. We deduce that

$$P[\sum_{\mu=2}^M S_1^\mu(J) < -\gamma_1 C_1 \lambda_1 |J|] \leq \exp\left(-\frac{\gamma_1 C_1}{2} \frac{\rho' N}{2\sqrt{\rho'} + 2\frac{\kappa}{\lambda_1} + \frac{Mm}{\gamma_1 C_1 (\lambda_1)^2}}\right).$$

There are  $\binom{N}{|J|}$  ways to choose the set  $J$ , and by Stirling's formula  $\binom{N}{|J|} \leq \exp(h(\rho')N)$ , where  $h(x) = -x \log x - (1-x) \log(1-x)$  is the entropy function introduced above. Using  $h(\rho') \leq -2\rho' \log(\rho')$ , we obtain the condition:

$$\frac{\gamma_1 C_1}{4} \frac{1}{2\sqrt{\rho'} + 2\frac{\kappa}{\lambda_1} + \frac{Mm}{\gamma_1 C_1 (\lambda_1)^2}} + \log(\rho') > 0,$$

which is true if

$$\frac{\gamma_1 C_1}{8} \frac{1}{2\sqrt{\rho'}} + \log(\rho') > 0, \tag{5.3}$$

as well as

$$\frac{\gamma_1 C_1}{8} \frac{\lambda_1}{2\kappa + \frac{Mm}{\gamma_1 C_1 \lambda_1}} + \log(\rho') > 0. \tag{5.4}$$

Now, there exists a  $\rho_0 \in (0, 0.1)$ , such that the first condition (5.3) is true if  $\rho' < \rho_0$ . The second condition (5.4) is true if

$$\rho' > \exp(-c \frac{\lambda_1}{2\kappa + \frac{Mm}{\gamma_1 C_1 \lambda_1}}),$$

where  $c = \frac{\gamma_1 C_1}{8}$ . This implies that, if there exists a constant  $c_2 > 0$  such that

$$\rho' \geq \rho_m := \exp(-c_2 \frac{\lambda_1}{\kappa + \frac{Mm}{\lambda_1}}),$$

then (5.4) is true.

For the second term, we have

$$\begin{aligned} & P[\exists I, |I| = \rho N, \exists J, |J| = \rho' N, \sum_{\mu=2}^M S_2^\mu(J, I) > \gamma_2 S^1(J, I)] \\ & \leq \sum_{I: |I|=\rho N} \sum_{J: |J|=\rho' N} P[\sum_{\mu=2}^M S_2^\mu(J, I) > \gamma_2 C_1 \lambda_1 |J|] \\ & \leq \sum_{I: |I|=\rho N} \sum_{J: |J|=\rho' N} \exp\left(-\frac{1}{2} \frac{\gamma_2 C_1 \lambda_1 |J|}{\lambda_{\{J, I\}} + \frac{Me\{J, I\}}{\gamma_2 C_1 \lambda_1 |J|}}\right), \end{aligned}$$

where  $\lambda_{\{J, I\}} = \lambda_1(E\{J, I\})$  is the largest eigenvalue of the graph determined by the undirected edges in  $E\{J, I\}$ . Using Lemma 4.3, we get

$$\lambda_{\{J, I\}} \leq 2[\sqrt{\rho\rho'}\lambda_1 + \kappa],$$

and  $e\{J, I\} \leq (\rho\rho'\lambda_1 + \sqrt{\rho\rho'}\kappa)N$ , which implies

$$P[\sum_{\mu=2}^M S_2^\mu(J) > \gamma_2 C_1 \lambda_1 |J|] \leq \exp\left(-\frac{\gamma_2 C_1}{2} \frac{\rho' N}{2\sqrt{\rho\rho'} + 2\frac{\kappa}{\lambda_1} + \frac{M\rho}{\gamma_2 C_1 \lambda_1} + \frac{M\kappa}{\gamma_2 C_1 (\lambda_1)^2} \sqrt{\frac{\rho}{\rho'}}}\right).$$

There are  $\binom{N}{|I|} \binom{N}{|J|}$  ways to choose the sets  $I$  and  $J$  and

$$\binom{N}{|I|} \binom{N}{|J|} \leq \exp((h(\rho) + h(\rho'))N) \leq \exp(2h(\rho)n),$$

as we assume that  $\rho' \leq \rho \leq 1/2$ . These considerations yield the condition

$$\frac{\gamma_2 C_1}{2} \frac{\rho'}{2\sqrt{\rho\rho'} + 2\frac{\kappa}{\lambda_1} + \frac{M\rho}{\gamma_2 C_1 \lambda_1} + \frac{M\kappa}{\gamma_2 C_1 (\lambda_1)^2} \sqrt{\frac{\rho}{\rho'}}} > 2h(\rho),$$

which is true if

$$\gamma_2 C_1 \frac{\rho'}{4\sqrt{\rho\rho'}} > 8h(\rho), \quad \gamma_2 C_1 \frac{\rho'}{4\frac{\kappa}{\lambda_1}} > 8h(\rho), \quad \frac{\gamma_2 C_1}{2} \frac{\rho'}{\frac{M\rho}{\gamma_2 C_1 \lambda_1}} > 8h(\rho),$$

and

$$\frac{\gamma_2 C_1}{2} \frac{\rho'}{\frac{M\kappa}{\gamma_2 C_1 (\lambda_1)^2} \sqrt{\frac{\rho}{\rho'}}} > 16\rho \log\left(\frac{1}{\rho}\right) \geq 8h(\rho).$$

From here we obtain the four conditions

$$\rho' > C^2 \rho h(\rho)^2, \quad \rho' > C \frac{\kappa}{\lambda_1} h(\rho), \quad \rho' > C' \frac{M}{\lambda_1} \rho h(\rho)$$

and

$$\rho' \geq \rho \left( \frac{2}{C'} \frac{M\kappa}{(\lambda_1)^2} \log\left(\frac{1}{\rho}\right) \right)^{2/3},$$

where  $C = \frac{32}{\gamma_2 C_1}$  and  $C' = \frac{16}{(\gamma_2 C_1)^2}$ .

For the third term, we have

$$\begin{aligned} & P[\exists I, |I| = \rho N, \exists J, |J| = \rho' N, \sum_{\mu=2}^M S_3^\mu(J, J) < -\gamma_3 S^1(J, I)] \\ & \leq \sum_{J: |J| = \rho' N} P\left[\sum_{\mu=2}^M S_3^\mu(J, J) < -\gamma_3 C_1 \lambda_1 |J|\right] \\ & \leq \sum_{J: |J| = \rho' N} \exp\left(-\frac{1}{2} \frac{\gamma_3 C_1 \lambda_1 |J|}{\lambda_{\{J, J\}} + \frac{Me\{J, J\}}{\gamma_3 C_1 \lambda_1 |J|}}\right), \end{aligned}$$

where  $\lambda_{\{J, J\}} = \lambda_1(E\{J, J\})$  is the largest eigenvalue of the graph determined by the undirected edges in  $E\{J, J\}$ . Using Lemma 4.3, we have

$$\lambda_{\{J, J\}} \leq 2\rho' \lambda_1 + 2\kappa$$

and  $e\{J, J\} \leq (\rho' \lambda_1 + \kappa) \rho' N$ , and we obtain as for the previous terms

$$\exp\left(-\frac{1}{2} \frac{\gamma_3 C_1 \lambda_1 |J|}{\lambda_{\{J, J\}} + \frac{Me\{J, J\}}{\gamma_3 C_1 \lambda_1 |J|}}\right) \leq \exp\left(-\frac{\gamma_3 C_1}{2} \frac{\rho' N}{(2 + \frac{M}{\gamma_3 C_1 \lambda_1}) \rho' + \frac{\kappa}{\lambda_1} (2 + \frac{M}{\gamma_3 C_1 \lambda_1})}\right).$$

There are  $\binom{N}{|J|}$  ways to choose the set  $J$ . From this, we get the condition

$$\frac{\gamma_3 C_1}{2(2 + \frac{M}{\gamma_3 C_1 \lambda_1})} \frac{1}{1 + \frac{\kappa}{\lambda_1 \rho'}} > h(\rho'),$$

which is true if

$$h(\rho') < C \quad \text{and} \quad h(\rho') < C \rho' \frac{\lambda_1}{\kappa}, \quad \text{where } C = \frac{\gamma_3 C_1}{4(2 + \frac{M}{\gamma_3 C_1 \lambda_1})}. \quad (5.5)$$

As we assume that  $M \leq c\lambda_1$ , there exists a  $\rho_1(\gamma_3, C_1) \in (0, 0.1)$ , such that the first inequality in (5.5) is true if  $\rho' < \rho_1$ .

Using the bound  $h(\rho') \leq -2\rho' \log(\rho')$  again, we get that there exists  $c > 0$  such that the second condition in (5.5) is true if

$$\rho' > \exp\left(-c \frac{\lambda_1}{\kappa}\right).$$

For the fourth term, we have

$$\begin{aligned}
& P[\exists I, |I| = \rho N, \exists J, |J| = \rho' N, \sum_{\mu=2}^M S_4^\mu(J, I) > \gamma_4 S^1(J, I)] \\
& \leq \sum_{I: |I|=\rho N} \sum_{J: |J|=\rho' N} P[\sum_{\mu=2}^M S_4^\mu(J, I) > \gamma_4 C_1 \lambda_1 |J|] \\
& \leq \sum_{I: |I|=\rho N} \sum_{J: |J|=\rho' N} \exp(-\frac{1}{2} \frac{\gamma_4 C_1 \lambda_1 |J|}{\lambda_{J \cap I} + \frac{Me\{J \cap I, J \cap I\}}{\gamma_4 C_1 \lambda_1 |J|}}),
\end{aligned}$$

where  $\lambda_{J \cap I} = \lambda_1(E\{J \cap I, J \cap I\})$  is the largest eigenvalue of the graph determined by the undirected edges in  $E\{J \cap I, J \cap I\}$ .

Using Lemma 4.3 and assuming that  $\rho' \leq \rho$ , we have  $\lambda_{J \cap I} \leq 2\rho' \lambda_1 + 2\kappa$  and  $e\{J \cap I, J \cap I\} \leq (\rho' \lambda_1 + \kappa) \rho' N$ , which are the same bounds as for the third term. There are  $\binom{N}{|I|} \binom{N}{|J|}$  ways to choose the sets  $I$  and  $J$  and using again

$$\binom{N}{|I|} \binom{N}{|J|} \leq \exp((h(\rho) + h(\rho')N) \leq \exp(2h(\rho)N),$$

we finally arrive at the same conditions as for the third term, with possibly a different constant  $C$ .

Finally, the different conditions can be summarized as:

$$\begin{aligned}
\rho' & \geq c_2 \rho \left(\frac{\kappa}{\lambda_1}\right)^2, & \rho_0 & \geq \rho \geq \rho' \geq \rho_m, & \rho' & > C^2 \rho h(\rho)^2, \\
\rho' & > C \frac{\kappa}{\lambda_1} h(\rho), & \rho' & > C' \frac{M}{\lambda_1} \rho h(\rho) & \text{and } \rho' & \geq \rho \left(\frac{2}{C'} \frac{M \kappa}{(\lambda_1)^2} \log\left(\frac{1}{\rho}\right)\right)^{2/3}.
\end{aligned}$$

Finally, taking into account all the conditions, we get that (5.1) is true if we choose

$$f(\rho) = \max\{c_1 \rho \left(\frac{\kappa}{\lambda_1}\right)^2, c_1 \rho h(\rho), c_1 \frac{\kappa}{\lambda_1} h(\rho), c_1 \rho \left(\frac{M \kappa}{(\lambda_1)^2} \log\left(\frac{1}{\rho}\right)\right)^{2/3}, \rho_m\}$$

for some  $c_1 > 0$  large enough and we see that  $f(\rho) \leq \rho$  if  $\rho \in (\rho_m, \rho_0)$  with  $\rho_0$  small enough.

□

In order to prove the Theorem 3.1, we will apply Lemma 5.1 repeatedly until the system attains an original pattern. Using

$$\rho_m = \exp(-c_2 \frac{\lambda_1}{\kappa + \frac{Mm}{\lambda_1}}),$$

we get that the system can attain an original pattern, i.e.  $\rho_m N < 1$ , only if

$$\kappa + \frac{Mm}{\lambda_1} < c_2 \lambda_1 / \log(N).$$

To determine the maximal number of steps the synchronous dynamics needs to converge, we analyze the following sequences :



**Lemma 5.2.** *Let  $(w_n)_{n \in \mathbb{N}}, (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$  and  $(z_n)_{n \in \mathbb{N}}$  such that*

$$w_0 = x_0 = y_0 = z_0 = \rho \in [\exp(-\frac{1}{2c} \frac{\lambda_1}{\kappa}), 1/e]$$

and

$$\begin{aligned} w_{n+1} &= cw_n \left(\frac{\kappa}{\lambda_1}\right)^2, & x_{n+1} &= cx_n h(x_n), \\ y_{n+1} &= c \frac{\kappa}{\lambda_1} h(y_n) & \text{and} \\ z_{n+1} &= cz_n \left(\frac{M\kappa}{(\lambda_1)^2} \log\left(\frac{1}{z_n}\right)\right)^{2/3}, \end{aligned}$$

for  $n \in \mathbb{N}$  and  $c > 0$ . Let us assume that  $\frac{\lambda_1}{\kappa} > C_1 \log N$  for some  $C_1 > 1$  large enough and that  $M \leq C_2 \lambda_1$  for some  $C_2 > 0$ . Then the sequences  $(w_n), (x_n), (y_n)$  and  $(z_n)$  are decreasing and there exists  $C_3 > 0$  and

$$n_0 \geq C_3 \max\left\{\log \log N, \frac{\log(N)}{\log\left(\frac{\lambda_1}{\kappa \log N}\right)}\right\}$$

such that  $\max\{w_{n_0}, x_{n_0}, y_{n_0}, z_{n_0}\} < 1/N$ .

*Proof.* Let us first consider the sequence  $(w_n)$ . Iterating  $w_{n+1} = aw_n$ , with  $a = c(\frac{\kappa}{\lambda_1})^2$ , we get trivially  $w_n = a^n w_0$  from which we deduce that  $w_n < \frac{1}{N}$  as soon as  $n > c_1 \frac{\log(N)}{\log(\frac{\lambda_1}{\kappa})}$  for some  $c_1 > 0$ .

For the sequence  $(x_n)$ , using  $h(x) \leq -2x \log(x) \leq 2\sqrt{x}$  for  $x \in [0, 1/2]$ , we have  $x_{n+1} \leq (Cx_n)^{3/2}$  for some constant  $C > 0$ . Iterating, we get  $x_n \leq (C^3 x_0)^{(3/2)^n}$ , from which we deduce that  $x_n < \frac{1}{N}$  if  $n \geq c_2 \log \log N$  for some  $c_2 > 0$ , if  $x_0$  is small enough.

For the sequence  $(y_n)$ , using again  $h(x) \leq -2x \log(x)$ , we have to iterate the relation  $y_{n+1} = a y_n \log(\frac{1}{y_n})$ , with  $a = 2c \frac{\kappa}{\lambda_1}$ . If we consider  $y_0 \in [\exp(-1/a), \exp(-1)]$ , the inductively defined sequence  $y_{n+1} = g(y_n)$  is decreasing and converges to  $\exp(-1/a)$  since the function  $g(x) = -ax \log(x)$  is increasing on the interval  $[\exp(-1/a), \exp(-1)]$ ,  $y_1 \leq y_0$  and  $\exp(-1/a)$  is the single fixed point of  $g$ . Moreover, we have

$$\begin{aligned} y_{n+2} &= a^2 y_n \log\left(\frac{1}{y_n}\right) \left(\log\left(\frac{1}{y_n}\right) + \log\left(\frac{1}{a}\right) + \log\left(\frac{1}{\log\left(\frac{1}{y_n}\right)}\right)\right) \\ &\leq a^2 y_n \log\left(\frac{1}{y_n}\right) \left(\log\left(\frac{1}{y_n}\right) + \log\left(\frac{1}{a}\right)\right), \end{aligned}$$

if  $y_n \leq 1/e$ . By iteration, if we set  $b = \log(\frac{1}{\min\{\rho, a\}})$ , we get similarly for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} y_n &\leq a^n y_0 \prod_{i=0}^{n-1} \left[ \log\left(\frac{1}{y_0}\right) + i \log\left(\frac{1}{a}\right) \right] \\ &\leq (ab)^n y_0 n! \\ &\leq c_3 (ab \frac{n}{e})^n \sqrt{n} \\ &= c_3 \exp(n(\log(a) + \log(b) + \log(n) - 1) + \frac{1}{2} \log(n)) \\ &\leq c_3 \exp(-c_4 \log N(1 + o(1))), \end{aligned}$$

for some  $c_3 > 0$ , if  $n = c_4 \log(N)/(-\log a - \log \log N)$ . In particular, this justifies the hypothesis  $\frac{\lambda_1}{\kappa} > c_5 \log N$  for some  $c_5 > 1$  large enough. We therefore see that there exists some  $c_6 > 0$  such that  $e^{-1/a} \leq y_n < \frac{1}{N}$  for  $n = c_6 \frac{\log(N)}{\log(\frac{\lambda_1}{\kappa \log N})}$ .

The third sequence can be rewritten as  $z_{n+1} = a z_n (\log \frac{1}{z_n})^{2/3}$ , with  $a = c(\frac{M}{\lambda_1} \frac{\kappa}{\lambda_1})^{2/3}$ . With the same technique as for  $y_n$ , we get that the sequence  $(z_n)$  converges to  $\exp(-1/a^{3/2})$  and  $z_n < 1/N$  if

$$n \geq c_7 \log(N) / \log\left(\frac{(\lambda_1)^2}{M \kappa \log(N)}\right).$$

This proves the lemma.  $\square$

The combination of the previous considerations and Lemma 5.2 then yields the Theorem 3.1.

#### APPENDIX A. ON THE DEGREES OF THE ERDÖS-RENYI GRAPH

To prove the corollary 3.3, we need to estimate the minimum and the maximum degrees of a typical Erdős-Renyi graph  $G(N, p)$ . The following result could not be found in the literature. We prove in this appendix :

**Lemma A.1.** *If  $G$  be is chosen randomly according to the model  $G(N, p)$ , then, if  $p \gg \frac{\log N}{N}$ , for a set of realizations of  $G$  the probability of which converges to one as  $N \rightarrow \infty$ , we have  $m = (1 + o(1))Np$  and  $\delta = (1 + o(1))Np$ .*

*Proof.* Let  $G$  chosen randomly according to the model  $G(N, p)$ . The law of the degree  $d_i$  of an arbitrary vertex  $i$  of  $G$  is the binomial distribution  $B(N, p)$ . Hence using the exponential Markov inequality, we arrive at the following bound : for  $p < a < 1$ , and  $N \geq 1$ ,

$$P[d_i \geq aN] \leq \exp(-NH(a, p)),$$

where  $H$  is the relative entropy or Kullback-Leibler information

$$H(a, p) = a \log\left(\frac{a}{p}\right) + (1 - a) \log\left(\frac{1 - a}{1 - p}\right).$$

If we now set  $m = \max_i d_i$  as above, we obtain

$$P[m \geq aN] \leq \sum_{i=1}^N P[d_i \geq aN] \leq N \exp(-NH(a, p)).$$

If we choose  $a = (1 + \varepsilon)p$ , for some  $\varepsilon > 0$  such that  $a < 1$ , we therefore get

$$P[m \geq (1 + \varepsilon)pN] \leq N \exp \left( -N(p(1 + \varepsilon) \log(1 + \varepsilon) + (1 - (1 + \varepsilon)p) \log(\frac{1 - (1 + \varepsilon)p}{1 - p})) \right).$$

Moreover we have  $(1 - (1 + \varepsilon)p) \log(\frac{1 - (1 + \varepsilon)p}{1 - p}) \geq -p\varepsilon$ .

Indeed, if we set  $q = 1 - p$  and  $u = 1 - (1 + \varepsilon)p = q - \varepsilon p$ , the last inequality is equivalent to  $\log(q/u) \leq q/u - 1$  which is true since  $q/u > 1$ . Thus

$$P[m \geq (1 + \varepsilon)pN] \leq N \exp(-Np((1 + \varepsilon) \log(1 + \varepsilon) - \varepsilon)) \leq N \exp(-Np \frac{\varepsilon^2}{2} (1 + o(1))),$$

if we suppose that  $\varepsilon = o(1)$  as  $N \rightarrow \infty$ . Choosing  $\varepsilon = 2\sqrt{\frac{\log N}{pN}}$ , we have  $\varepsilon = o(1)$

for  $p \gg \frac{\log N}{N}$ , and

$$P[m \geq (1 + \varepsilon)pN] \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Moreover, we have  $m \geq \lambda_1$  and  $\lambda_1 = (1 + o(1))pN$  (with probability converging to 1 as  $N \rightarrow \infty$ ), which gives the result for  $m$ .

Now, if we set  $\delta := \min_i d_i$ , we want to prove that  $P[\delta \geq (1 + \varepsilon')pN] \rightarrow 1$ , as  $N \rightarrow \infty$ , for some  $\varepsilon' = o(1)$ . We consider the complementary graph, i.e. the random graph  $\overline{G}$ , such that exactly those edges are missing in a realization of  $\overline{G}$ , that occur in the corresponding realization of the original random graph  $G$ . Now the maximum degree  $\overline{m}$  of  $\overline{G}$  and the minimum degree  $\delta$  of  $G$  are linked via the relation  $\delta = N - 1 - \overline{m}$ .

As  $\overline{G}$  is chosen randomly according to the model  $G(N, 1 - p)$ , we have

$$P[\overline{m} \geq (1 + \varepsilon)(1 - p)N] \leq N \exp(-NH((1 + \varepsilon)(1 - p), 1 - p)),$$

for all  $\varepsilon > 0$  such that  $(1 + \varepsilon)(1 - p) < 1$ . Now

$$H((1 + \varepsilon)(1 - p), 1 - p) = (1 + \varepsilon)(1 - p) \log(1 + \varepsilon) + (p - \varepsilon + p\varepsilon) \log(1 - \frac{\varepsilon(1 - p)}{p}).$$

If we suppose that  $\varepsilon = o(1)$  and  $\varepsilon \ll p$ , using the inequality  $\log(1 - x) \geq -x - x^2/2 - x^3$  for  $x \in (0, 1/2)$  to bound the last term, we obtain the estimate

$$H((1 + \varepsilon)(1 - p), 1 - p) \geq \frac{\varepsilon^2}{2p}((1 - p) - C(\varepsilon + \frac{\varepsilon}{p}) + \mathcal{O}(p\varepsilon^2)),$$

for some  $C > 0$  and

$$P[\overline{m} \geq (1 + \varepsilon)(1 - p)N] \leq \exp(-N \frac{\varepsilon^2}{2p}((1 - p) - C(\varepsilon + \frac{\varepsilon}{p}) + \mathcal{O}(p\varepsilon^2)) + \log(N)).$$

There exists some  $c > 0$  such that if we choose  $\varepsilon = \sqrt{\frac{4p}{c(1-p)} \frac{\log(N)}{N}}$ , we get

$$P[\bar{m} \geq (1 + \varepsilon)(1 - p)N] \leq \exp(-cN \frac{\varepsilon^2}{2p}(1 - p) + \log(N)) \rightarrow 0,$$

under the conditions  $p \gg \frac{\log N}{N}$  and  $1 - p \gg (\frac{\log N}{N})^{1/3}$ .

Finally we get  $\delta \geq N - 1 - (1 + \varepsilon)(1 - p)N = (1 + o(1))Np$ , which is the result under these two conditions.

Eventually, we will extend this result for all  $p$  such that  $p \rightarrow 1$ , as  $N \rightarrow +\infty$ . As previously, using the exponential Markov inequality, we get the following bound : for  $0 < b < p < 1$ , and  $N \geq 1$ ,

$$P[d_i \leq bN] \leq \exp(-NH(b, p)).$$

We set  $p = 1 - a_N$  and  $b = 1 - b_N$ , for some strictly positive sequences  $(a_N)$  and  $(b_N)$  such that  $a_N + b_N \rightarrow 0$ , as  $N \rightarrow \infty$ ,  $a_N \ll b_N$ , and we can restrict to the case  $a_N < (c \frac{\log N}{N})^{1/3}$  for some  $c > 0$ . We get

$$\begin{aligned} P[\delta \leq bN] &\leq N \exp(-N((1 - b_N) \log(\frac{1-b_N}{1-a_N}) + b_N \log(\frac{b_N}{a_N}))) \\ &\leq \exp(-N(b_N \log(\frac{b_N}{a_N}) - 2b_N) + \log(N)) \end{aligned}$$

So, we need to choose  $b_N$  such that

$$b_N \log(\frac{b_N}{a_N}) > \frac{\log(N)}{N}.$$

We have

$$b_N \log(\frac{b_N}{a_N}) > b_N \log(b_N (\frac{N}{c \log(N)})^{1/3}) > \frac{\log(N)}{N},$$

if we choose for instance  $b_N = (\frac{\log N}{N})^\gamma$  with  $\gamma \in (0, 1/3)$ .

Finally, we get for all  $p \rightarrow 1$  that  $\delta \geq (1 - b_N)N = (1 + o(1))Np$ , with probability converging to 1 as  $N \rightarrow \infty$ .  $\square$

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